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EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

AND

PROF. E. T. WHITTAKER, M.A., F.R.S.

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VOL. X. MAY, 1920. No. 146.

THE GRAPHICAL TREATMENT OF DIFFERENTIAL EQUATIONS.

BY DR. S. BRODETSKY.

(*Continued from p.* 38.)

7. The method to follow when we have more than one family of non-intersecting curves is at once clear. We consider each family in turn, and then superpose the results in order to get the complete solution. We shall at once proceed to the consideration of a few examples. We shall restrict ourselves to the case of irreducible equations, as the reducible ones are of no particular interest.

(i) When we have $(a^2-x^2)y_1{}^2-x^2=0$, we have either

$$y_1=x/(a^2-x^2)^{\frac{1}{2}} \quad \text{or} \quad y_1=-x/(a^2-x^2)^{\frac{1}{2}},$$

in which $(a^2-x^2)^{\frac{1}{2}}$ is to be interpreted arithmetically. We now introduce the well-known notation

$$p\equiv\frac{dy}{dx},$$

and distinguish the two families thus:

$$p_1=\frac{x}{(a^2-x^2)^{\frac{1}{2}}}, \quad p_2=-\frac{x}{(a^2-x^2)^{\frac{1}{2}}}.$$

We at once show that $$\frac{dp_1}{dx}=\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}},$$

and is therefore always positive since $(a^2-x^2)^{\frac{1}{2}}$ is to be interpreted arithmetically. It is easily seen that the family p_1 consists of the semi-circles in Fig. 12 (a); similarly the family p_2 consists of the semi-circles in Fig. 12 (b). The analytical forms can be found in this case, and they are

$$y=-(a^2-x^2)^{\frac{1}{2}}+C$$

and $$y=(a^2-x^2)^{\frac{1}{2}}+C,$$

where C is an arbitrary constant and the square root is interpreted arithmetically. The superposition of (a) and (b) gives the complete solution, 12 (c) consisting of complete circles. But each circle consists of one half (the lower)

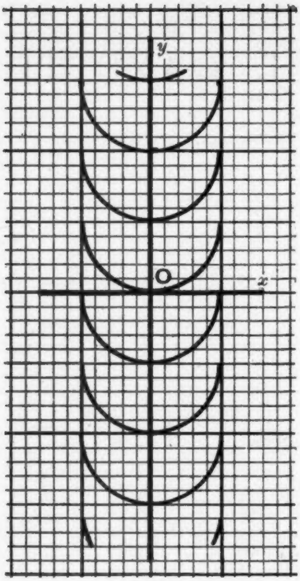

Fig. 12 (a).

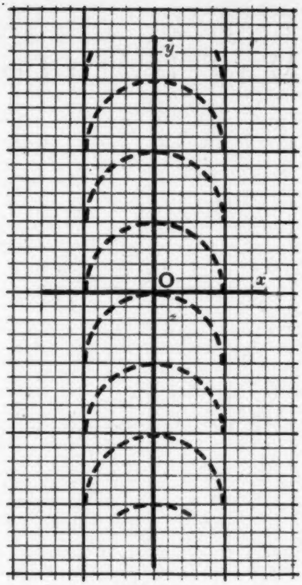

Fig. 12 (b).

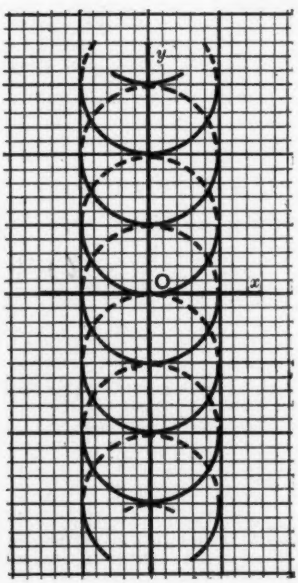

Fig. 12 (c).

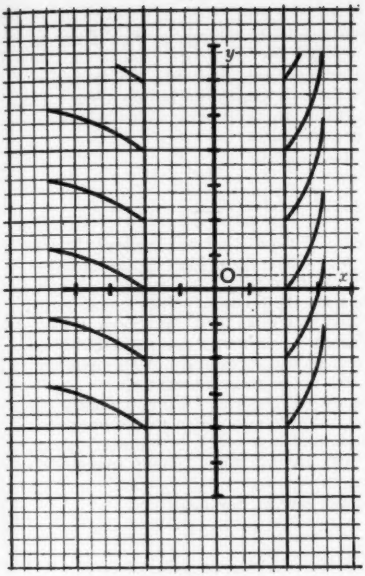

Fig. 13 (*a*).

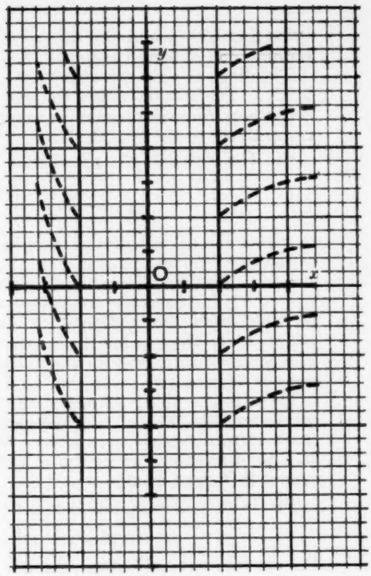

Fig. 13 (*b*).

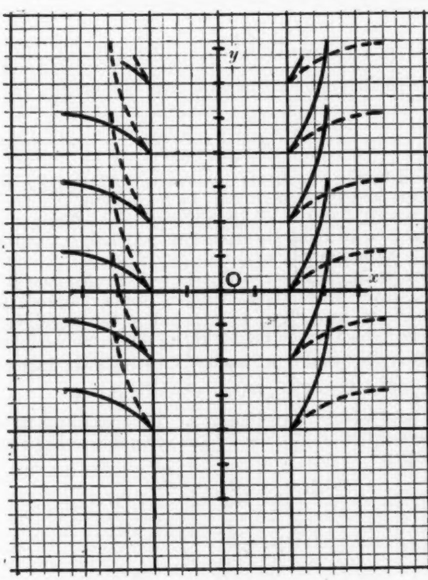

Fig. 13 (*c*).

belonging to p_1, and the other half (the upper) belonging to p_2. There is nothing real outside $x = \pm a$.

The lines $x = \pm a$ are a singular solution, and are obviously the envelope locus. The way in which the envelope locus is generated is now clear: it consists of elements of common tangents at the junctions of members belonging respectively to the two families. $x = 0$ is a tac locus.

(ii) The equation $p^2 - xp + 1 = 0$ gives two families

$$p_1 = \frac{x + (x^2 - 4)^{\frac{1}{2}}}{2}, \quad p_2 = \frac{x - (x^2 - 4)^{\frac{1}{2}}}{2},$$

in which the square root is arithmetical. We find that the general shapes of the two families are as in Fig. 13 (a), (b). There is nothing inside the lines $x = \pm 2$. The curves are as drawn. At the boundary lines $x = \pm 2$, we get $p_1 = p_2 = 1$, but each one is not the same as for the boundary lines themselves, for which $dy/dx = \infty$. Hence $x = \pm 2$ represent a cusp locus, as shown in Fig. 13 (c), where the two families are superimposed to give the complete solution. The geometrical reason for the existence of a cusp locus and not an envelope locus is thus clearly exhibited. $x = \pm 2$ are not a singular solution.

There is one disadvantage of the geometrical process. When a family of curves consists each of two branches, it may not be possible to discover the exact correspondence between one branch and the other. Symmetry does help us sometimes, as in the example just given. But in some cases this cannot be helped at all.

(iii) Take *e.g.* the equation $p^2 = x + y$. We get the families

$$p_1 = (x + y)^{\frac{1}{2}}, \quad p_2 = -(x + y)^{\frac{1}{2}}.$$

The curves exist only above the line $x + y = 0$. At this boundary $p_1 = p_2 = 0$, which is different from the value of dy/dx for the line itself. We therefore have a cusp locus, and $x + y = 0$ is not a singular solution.

We find $$\frac{dp_1}{dx} = \frac{1}{2}\left\{1 + \frac{1}{(x + y)^{\frac{1}{2}}}\right\}.$$

This is always positive, and the general form is as in Fig. 14 (a). But

$$\frac{dp_2}{dx} = \frac{1}{2}\left\{1 - \frac{1}{(x + y)^{\frac{1}{2}}}\right\}.$$

Hence the second family is concave up above the line $x + y = 1$, and convex up below this line, as exhibited in Fig. 14 (b). Along $x + y = 1$ we find that not only is $dp_2/dx = 0$, but also all the differential coefficients of p_2 are zero. We have, in fact, a straight-line solution, which is at the same time an asymptote to the second family of curves.

The complete solution is given in Fig. 14 (c): in this case it is apparently impossible to discover which branch above the asymptote in the second family corresponds to any given branch below the asymptote. Does the analytical solution (which *can* be found) throw much more light on this point?

(iv) An equation discussed in a standard treatise is the following:

$$p^2y^2 \cos^2\alpha - 2pxy \sin^2\alpha + y^2 - x^2 \sin^2\alpha = 0.$$

We solve for p and obtain two families. To enable us to use a definite geometrical process, let us take $\alpha = \pi/4$: the general case has exactly the same properties. We get

$$p_1 = \frac{x}{y} + \sqrt{2}\frac{(x^2 - y^2)^{\frac{1}{2}}}{y},$$

$$p_2 = \frac{x}{y} - \sqrt{2}\frac{(x^2 - y^2)^{\frac{1}{2}}}{y}.$$

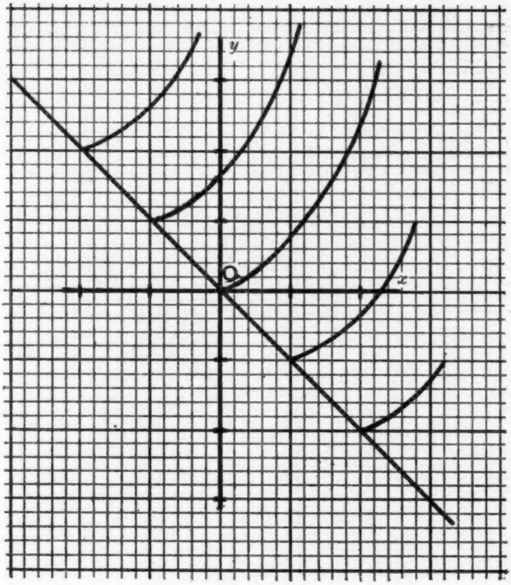

Fig. 14 (a).

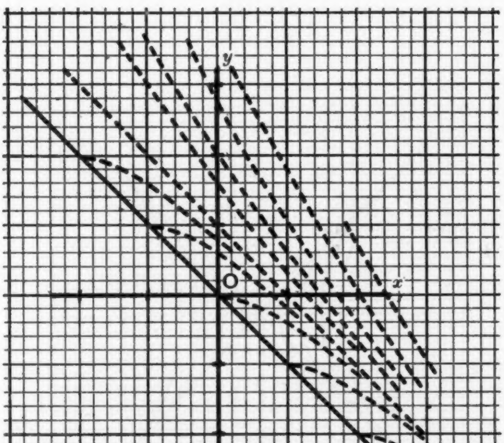

Fig. 14 (b).

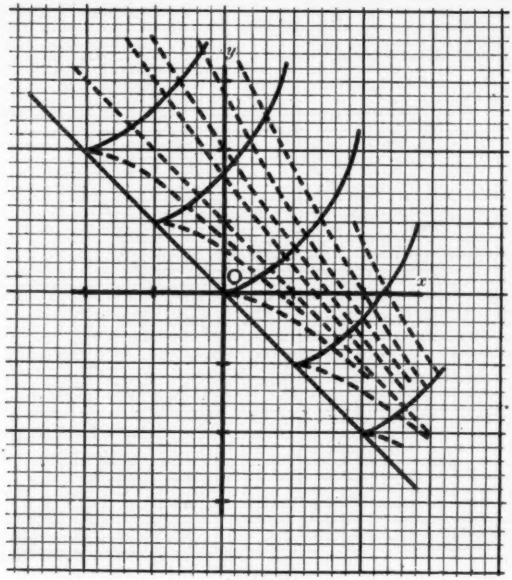

FIG. 14 (c).

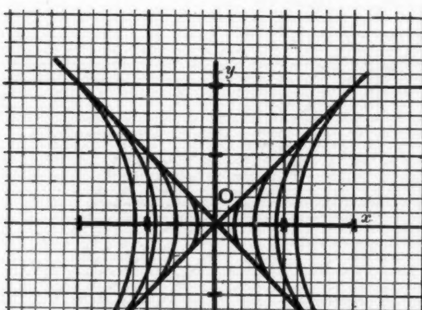

FIG. 15 (a).

The curves exist only within 45° of the axis of x. We get the general forms shown in Figs. 15 (a), (b), and the actual curves are easily found to be as sketched. Family I. consists of similar arcs of circles, with centre of similitude at the origin. Family II. consists of the remainders of the circles. When we

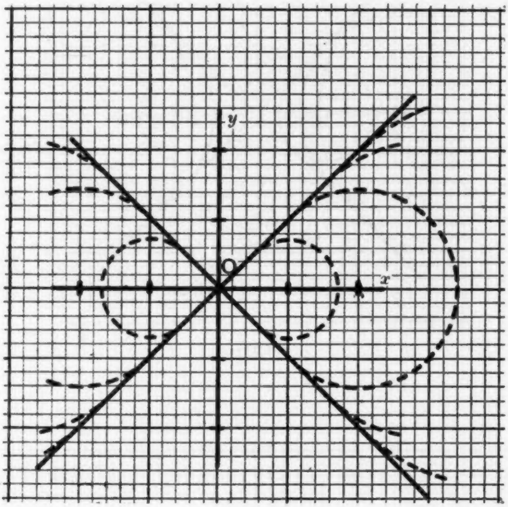

FIG. 15 (b).

superimpose as in Fig. 15 (c), we get complete circles. The lines $y = \pm x$ are envelopes, and therefore the singular solution. $y=0$ is a tac locus. The verification of these details is left to the reader, who is also advised to discuss the general value of a.

(v) The equation $(x^2 - a^2)p^2 - 2xyp - x^2 = 0$ gives two families:

$$p_1 = \frac{x\{y + (x^2 + y^2 - a^2)^{\frac{1}{2}}\}}{x^2 - a^2},$$

$$p_2 = \frac{x\{y - (x^2 + y^2 - a^2)^{\frac{1}{2}}\}}{x^2 - a^2};$$

there are no curves inside the circle $x^2 + y^2 = a^2$.

If we work out $\frac{dp_1}{dx}$ and $\frac{dp_2}{dx}$, we get the general forms shown in Figs. 16 (a), (b), and a consideration of some directions of tangents at once leads to the curves drawn: the two families are symmetrical about the axis of y. The lines $x = \pm a$ are straight-line solutions. The boundary $x^2 + y^2 = a^2$ is an envelope locus, because at any point on this circle $p_1 = p_2 = \frac{xy}{x^2 - a^2}$, and this is the same value as is obtained for dy/dx for the boundary itself. $x=0$ is a tac locus; $y=0$ is a node locus. The complete solution is given in Fig. 16 (c).

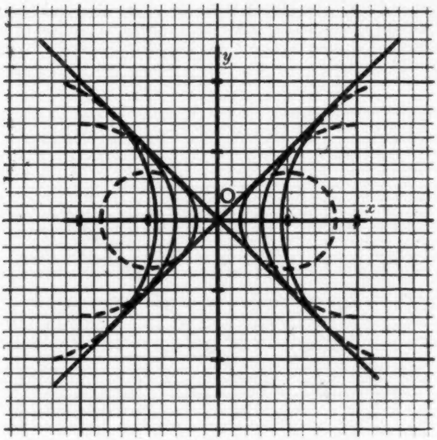

Fig. 15 (c)

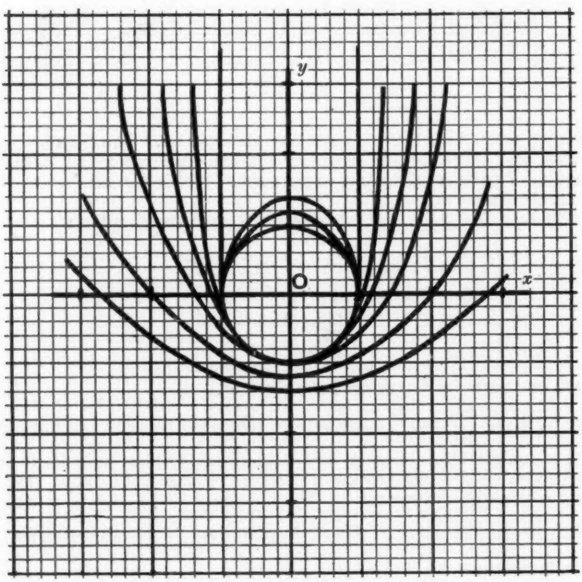

Fig. 16 (a).

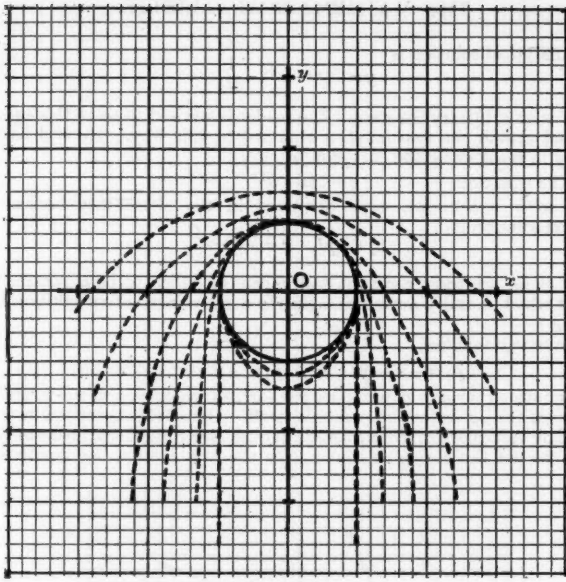

Fig. 16 (b).

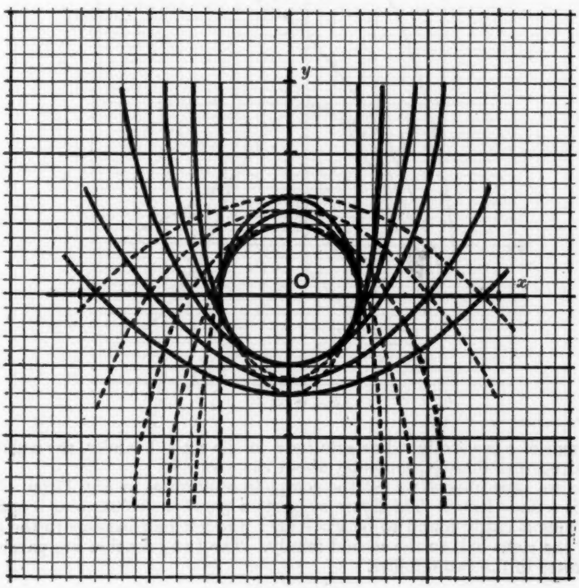

Fig. 16 (c).

The analytical solution is

$$y=C(x^2-a^2)-\frac{1}{4C},$$

where C is an arbitrary constant. The reader is advised to solve this equation for C and sketch the two families obtained in this way. He will have no difficulty in recognising the curves in (*a*), (*b*).

(vi) The equation $$y=2px+p^2$$
belongs to a well-known type discussed in the elementary books. The analytical solution is

$$y=2px+p^2, \quad x=-\tfrac{2}{3}p+\frac{C}{p^2},$$

where C is an arbitrary constant and p can be taken to be a parameter in terms of which x and y are given. Surely a more instructive solution is the following ?

We have two families,

$$p_1=-x+(x^2+y)^{\frac{1}{2}}, \quad p_2=-x-(x^2+y)^{\frac{1}{2}}.$$

There is nothing inside the parabola $y=-x^2$, at which $p_1=p_2=-x$, and this differs from the dy/dx for the parabola itself. Hence this boundary is a cusp locus. The values of dp_1/dx and dp_2/dx at once give the general forms in 17 (*a*), (*b*), and we get the curves as drawn. In the first family, the positive half

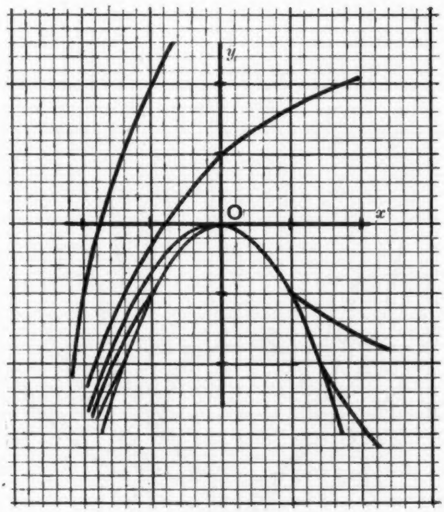

FIG. 17 (*a*).

of the axis of x is a straight-line solution; in the second family, the negative half: altogether there is symmetry about the y-axis.

When we superimpose we get the complete solution as in Fig. 17 (*c*). The y-axis is a node locus. The figure suggests a parabolic solution as a special case. This is easily found to be $y=-\frac{3}{4}x^2$; the left half belongs to the first family, the right half to the second family. The two branches are parabolic asymptotes to the two families.

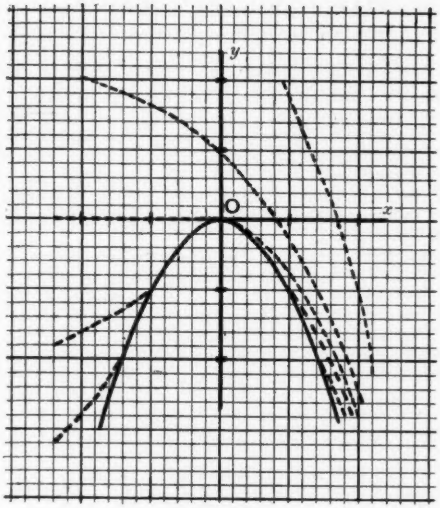

FIG. 17 (b).

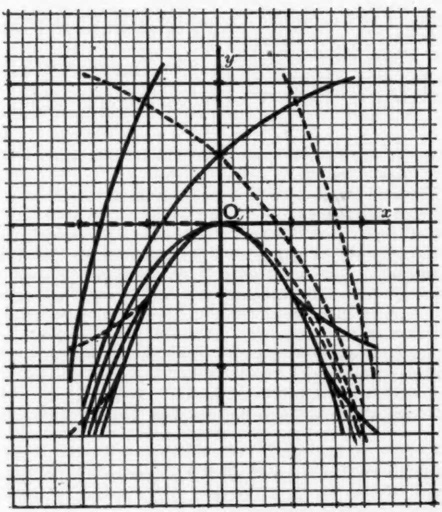

FIG. 17 (c).

RATIO AND PROPORTION.

By D. K. Picken, M.A.

(*Continued from p.* 13, *January*, 1920.)

6. There remains the further extension to what may be called "multiplicate proportion" (duplicate, triplicate, ... n-plicate). This may be regarded as a limiting case of compound proportion, being a relation between variable quantities P, X of two different kinds such that *measures* p, x of these variables are in the proportional relation $p \propto x^n$. (The definition can, of course, be expressed—like the definitions of direct and inverse proportion—in terms of ratios of any two quantities of the one kind and the two corresponding quantities of the other kind.)

E.g. The volume of a rectangular solid is jointly proportional to the lengths of three edges; but if we restrict the three lengths to be equal, and therefore necessarily to vary *together*, we have a case of multiplicate proportion, viz. the volume of a *cube* is in triplicate proportion to the length of its edge. (From this result, on a principle which has been stated above, we have the name "cm^3" for the standard unit of volume; but it should be noted that we can state precisely analogous propositions on the ellipsoid and sphere without being at liberty to use the term cm^3 also for the volume of the sphere of radius (or diameter) cm; it is most important to recognise the element of convention in the use of these "product" and "power" expressions in terms of *physical* quantities.)

The phrase "P varies as the nth power of X" is used; but it is (at any rate from the point of view of elementary teaching) an unfortunate phrase, since it encourages the assumption that there is a quantity which can properly be called "the nth power of X"—encourages, in fact, the disposition to treat the whole business of quantities as purely algebraic. There is for example no quantity which can properly be called "the square of a velocity"; but this is a type of fact certainly not clearly (however it may be vaguely) understood by the ordinary student.

7. Finally, a multiplicate proportion may be compounded with other proportions, simple or multiplicate.

An important case is the Law of Gravitation, giving the algebraic form

$$F \propto m \,.\, m'/d^2,$$

in which F; m, m'; d are *number*-quantities which have reference respectively to units of Force, Mass, Length.

A little further consideration of this case brings out points of importance in the application of the theory:—

All the "derived units" of Physical Science are "derived" from relationships of proportion, by the principle of "correspondence" of unit quantities in such relationships—as indicated above: in the most general type of case it is relationships of joint multiplicate proportion that are involved. The principle and process of derivation from fundamental proportional relations is so invariable, but at the same time so badly expressed, that it is apt to pass over the student's head unnoticed. The Law of Gravitation is, however, a relation involving (apparently) only three types of quantity, for which units have been otherwise determined in Dynamical theory. Hence (and, of course, similarly in all such cases) the simplest form of its algebraic expression is

$$F = G \,.\, m \,.\, m/d^2,$$

in which G is an *algebraic* quantity which is *constant with respect to the variables of the proportional relation but is a function of the system of units involved in the algebraic expression of that relation.*

The quantity G, thus defined (or, rather, defined in such a manner that this seems the only accurate way to interpret the definition) is commonly called "the constant of gravitation"; but as so defined it is an "algebraic constant", not a "physical constant", the value varying with the system of units employed.

It is, however, a right instinct* which regards "the constant of gravitation" as, in fact, an *absolute physical constant*, definite and invariable like the angle "of one revolution" or the angle *radian*.

The peculiarity of the case—a quite unique case in Physical Science—is the fact that "the constant of gravitation" so conceived is a physical quantity which is *the only one of its kind*. This might at first sight appear to be a contradiction in terms, seeing that the term *quantity* by its definition implies comparison (by the comparatives "greater" and "less") among different quantities of the same kind, as basis of the very conception of quantity. But the usage in the present connection is quite a proper one—as a species of "limiting case"; all mathematical terms get such logical extensions of use after they are properly established in the science. Here the justification is found in the analogy with the other cases in which the Inverse Square Law operates—the cases of Electrical and Magnetic action; and in the closely associated, and commonly recognised, scientific possibility that other quantities of the same kind as the physical constant of gravitation may in fact exist though they have not yet been discovered.

From this point of view the quantity G, as defined above, is to be regarded as *the measure* of the physical constant of gravitation, in terms of a "unit" inherent in the system of units employed: so that the abovementioned scientific possibility has to be "exploited" to provide such units.

But these somewhat obscure theoretical facts tend to be a serious obstacle to elementary recognition of the constant of grávitation as a physical quantity. And the difficulty is in fact quite unnecessary—as may be seen by a further consideration of the Electrical and Magnetic analogies. In those cases the unit of the kind in question (*inductivity* or *permeability*) is regarded as a fundamental unit of the system of Physical units; and the unit chosen is the quantity of the kind which actually occurs in the simplest instance, viz. that of the aether (or vacuous medium). The analogous procedure would be to take "the constant of gravitation" as unit of its own kind—than which, quite apart from any such argument, nothing could be more obvious common sense; here, if anywhere in the whole range of Physical Science, we have a "natural unit" which has no rival.

This procedure is, in fact, implicit in the unit-system of Astronomical Dynamics. In that system the unit of Mass is not one of the fundamental units, but is "derived" *along with the unit of Force* so that these two units may be quantities which "*correspond*" with one another and with the units of Length and Time in the *two* proportional relations which belong respectively to the Second Law of Motion and the Law of Gravitation. In the light of the above reasoning it seems, to the writer of this paper, an extraordinary anomaly that the Astronomical units of Mass and Force have not been incorporated in the general system of Physical Units.

[*Note*: This final illustration may serve to show how far-reaching is the Proportion theory with which this paper is primarily concerned: how it is the alphabet of the whole theory of Measurement, Units and Dimensions.

The unfamiliarity of much of the argument that has been employed can be attributed only to the way in which facts underlying Physical theory have been

* Right instinct, often the instinct of the greatest genius, combined with sketchy mathematics is characteristic of Physical writing. The blame lies not so much at the door of Physics as of Mathematics. Unfortunately the loose expression gets transmitted to the learner; frequently the accurate thought behind it does not, especially when the vehicle is not a great physicist but a book.

glossed over by loose mathematical expression—which meets the passing need of the busy practical physicist but is utterly unsatisfactory as a permanent medium. If anyone be disposed to assume that all this argument must be an unnecessary refinement of facts so generally apprehended in a much more direct and immediate way, it may be suggested to compare with such facts as the common apprehension of *length of a curved line* and the comparative abstruseness of its accurate mathematical specification.

The writer has hesitated long before attempting to publish these opinions—realising the difficulty of writing on such familiar questions in such a way as to carry conviction. But the great importance of the subject has impressed him only more forcibly with the passage of time—and the grave consequences, to Exact Science, of what he regards as very general failure to realise these facts in an absolutely clear-cut way. His object will be achieved if discussion should be stimulated; though it is a matter of regret to him that natural circumstances must prevent him from taking an effective part in any such discussion.] D. K. Picken.

The Lodge, Ormond College, University of Melbourne.

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43. It was a problem which he could not solve,
'Twas just what mathematics are to me,
A science which the longer I revolve,
The surer am I we shall ne'er agree . . .
My Simpson's Euclid, you're a cursed bore,
Although, no doubt, a treasure in your way,
And those who doat on science may explore
Your problems—with what appetite they may.
I have no head for mathematic lore,
Therefore, my Simpson's Euclid, I must say
(Though I'm desirous to not to be uncivil)
I must devoutly wish you at the Devil.
—Moultrie's *Godiva*, xxxi.-xxxii.

44. Comminuent. To avoid the tedious repetition of "a quantity which diminishes without limit when Δx diminishes without limit," I have coined this word. There are sufficient analogies for the derivation, or at any rate we must not want words because Cicero did not know the Differential Calculus. Hence we add to our dictionary as follows: To *comminute* two quantities, is to suppose them to diminish without limit together: *comminution*, the corresponding substantive, *comminuents*, quantities which diminish without limit together. To *comminute* has been used in the sense of to *pulverise*, and is therefore recognised English.—De Morgan, *Diff. and Integral Calculus* (1842), n.p., 66.

45. . . . . forgetful of the claims
Of curves and squares, and parallelograms,
Cones, angles, sines and cosines, ordinates,
Abscissae and the like.—Moultrie, *The Dream of Life*, i. 419.

46. "Das Leben der Götter ist Mathematik. Alle göttlichen Gesandten müssen Mathematiker sein. Reine Mathematik ist Religion. Die Mathematiker sind die einzig Glücklichen. Der echte Mathematiker ist Enthusiast *per se.* Ohne Enthusiasmus, keine Mathematik. Echte Mathematik ist das eigentliche Element des Magiers. In der Musik erscheint sie förmlich als Offenbarung, als schaffender Idealismus. Hier legitimiert sie sich als himmlische Gesandte, *kat'anthropon*. Das höchste Leben ist Mathematik. Zur Mathematik gelangt man nur durch eine Theophanie.

"Wer ein mathematisches Buch nicht mit Andacht ergreift und es wie Gottes Wort liest, der versteht es nicht."—Novalis, *Schriften*, p. 223 (Berlin, 1901, 2nd ed.). [Per H. G. Forder, M.A.]

MATHEMATICAL NOTES.

542. [U.] *Geocentric Parallax.*

[The following elementary method of obtaining the fundamental equations of geocentric parallax is directly by Spherical Trigonometry, and may be of interest to readers. The notation here followed is the same as that in Art. 93 of Sir Robert Ball's *Spherical Astronomy.*]

Let $Z(\vartheta, \phi')$, $S(\alpha, \delta)$, $S'(\alpha', \delta')$ be the points on the celestial sphere (Fig 2) to which the lines OO', OS, $O'S$ (Fig. 1) are severally directed; ρ the earth's

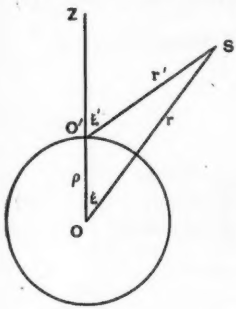


FIG. 1.

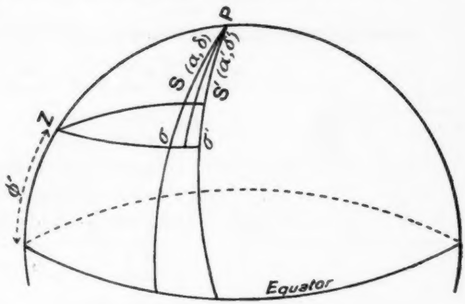


FIG. 2.

radius and r, r' the distances of the celestial body S from the earth's centre O, and the observer's position O', respectively; P the pole of the celestial sphere.

To find $\alpha' - \alpha$, the parallax in Right Ascension.

Let $\qquad P\hat{S}'Z = \epsilon, \quad Z\hat{O}S = \xi, \quad Z\hat{O}'S = \xi'.$

From the triangle $O'OS$ in Fig. 1, we have

$$\frac{\sin SS'}{\rho} = \frac{\sin S'Z}{r} = \frac{\sin SZ}{r'}; \qquad \text{.......................(1)}$$

and from the spherical triangles PSS' and PZS', we get

$$\frac{\sin SS'}{\sin(\alpha' - \alpha)} = \frac{\cos\delta}{\sin\epsilon}, \qquad \text{.......................(2)}$$

$$\frac{\sin ZS'}{\sin(\alpha' - \vartheta)} = \frac{\cos\phi'}{\sin\epsilon}. \qquad \text{.......................(3)}$$

Dividing (2) by (3) and substituting for the arcs from (1), we get

$$\frac{\rho}{r} \cdot \frac{\sin(\alpha' - \vartheta)}{\sin(\alpha' - \alpha)} = \frac{\cos\delta}{\cos\phi'},$$

$$\textit{i.e.} \quad \frac{\rho}{r}\{\cos(\alpha' - \vartheta) + \cot(\alpha' - \alpha)\sin(\alpha - \vartheta)\} = \frac{\cos\delta}{\cos\phi'};$$

whence we derive the formula

$$\tan(\alpha' - \alpha) = -\frac{\frac{\rho}{r}\cos\phi'\sin(\vartheta - \alpha)}{\cos\delta - \frac{\rho}{r}\cos\phi'\cos(\vartheta - \alpha)}. \qquad \text{.......................(A)}$$

To find $\delta' - \delta$, the parallax in Declination.

Draw $Z\sigma\sigma'$ perpendicular to the bisector of the angle SPS' to meet the declination circles of S, S' in σ, σ' respectively.

Evidently $$P\sigma = P\sigma'.$$

Let the declination of σ or σ' be γ.

From the usual relation between the arcs joining three points on a great circle and another point, we have

$$\cos PZ \sin SS' + \cos PS' \sin ZS = \cos PS \sin ZS',$$

$$\text{i.e.} \quad r \sin \delta - r' \sin \delta' = \rho\beta \sin \gamma, \text{ where } \beta = \frac{\sin \phi'}{\sin \gamma}. \quad \text{.............. (4)}$$

Again, from the spherical triangles $Z\sigma S$ and $Z\sigma' S'$, we get

$$\frac{\sin \sigma S}{\sin ZS} = \frac{\sin \sigma ZS}{\sin P\sigma Z} = \frac{\sin \sigma' S'}{\sin ZS'},$$

$$\text{i.e.} \quad \frac{\sin(\delta - \gamma)}{\sin(\delta' - \gamma)} = \frac{r'}{r},$$

$$\text{i.e.} \quad \sin \gamma (r \cos \delta - r' \cos \delta') = \cos \gamma (r \sin \delta - r' \sin \delta'),$$

$$\text{i.e.} \quad r \cos \delta - r' \cos \delta' = \rho\beta \cos \gamma. \quad \text{.................................. (5)}$$

Multiplying (4) by $\cos \delta'$ and (5) by $-\sin \delta'$ and adding, we get

$$\frac{\sin(\gamma - \delta')}{\sin(\delta - \delta')} = \frac{r}{\rho\beta},$$

$$\text{i.e.} \quad \sin(\gamma - \delta) \cot(\delta - \delta') + \cos(\gamma - \delta) = \frac{r}{\rho\beta},$$

from which we easily derive

$$\tan(\delta' - \delta) = \frac{\rho\beta \sin(\delta - \gamma)}{r - \rho\beta \cos(\delta - \gamma)}. \quad \text{.......................... (B)}$$

Mysore. A. A. KRISHNASWAMI AIYANGAR.

543. [R. X. 4.] *A Graphical Treatment of Simple Harmonic Motion.*

The importance of a treatment of Simple Harmonic Motion in any course of Mechanics will be granted at once by any reader of this note. Yet, so far as the author is aware, one of two methods is usually adopted, each having, in his opinion, serious disadvantages. The first, and perhaps more common, method makes S.H.M. depend upon the previously considered properties of a uniform circular motion, of which it is the projection. The two main objections to this are, first, that the fact of the acceleration being proportional to the displacement emerges only after a time, instead of being considered as the fundamental fact determining the motion; secondly, that there is usually serious confusion when these ideas are applied to the pendulum, since its bob moves in an arc of a circle. The other method, which consists in integrating the differential equation of motion, while mathematically unexceptionable, demands a much better knowledge of the principles and processes of the Calculus than can be expected of boys reaching the subject for the first time, probably (and preferably) in a course having a distinct physical bias. These considerations have led to a third method, explained below.

It is assumed that laboratory work will have supplied the main facts with regard to the simple pendulum, *i.e.* its isochronism, period $\propto \sqrt{\text{length}}$, restoring force $\propto$ displacement. (It is quite easy, and instructive, to show this last fact experimentally, afterwards pointing out its connection with the parallelogram of forces.) It is now necessary to show mathematically the interdependence of these facts.

If the study of kinematics has been based on graphical methods, as is now common, it will not be difficult to lead up to the idea of work being measured by an area on the force-displacement graph. In our case the graph is a straight line, so that it is easy to show that if $F=\mu x$ work done from position of instantaneous rest $(x=a)$ to the position where the displacement is x is, by the shaded area,

$$\frac{\mu}{2}(a^2-x^2).$$

The principle of energy now gives

$$\tfrac{1}{2}mv^2=\frac{\mu}{2}(a^2-x^2)\,; \quad \therefore \ v=\sqrt{\frac{\mu}{m}}\cdot\sqrt{a^2-x^2}.$$

Now plot the curve $y=\sqrt{a^2-x^2}$, *i.e.* a circle of radius a (see Fig. 2).

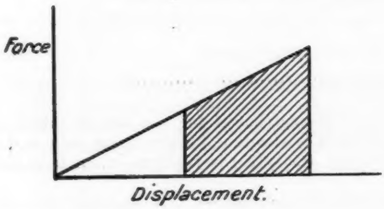


FIG. 1.

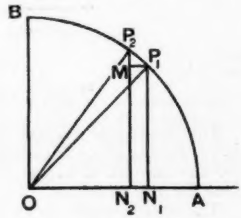


FIG. 2.

Now time taken to move from N_1 to N_2

$$=\frac{N_1N_2}{v}=\frac{N_1N_2}{P_1N_1\sqrt{\mu/m}}=\sqrt{\frac{m}{\mu}}\cdot\frac{N_1N_2}{P_1N_1}.$$

But, by similar triangles,

$$\frac{N_1N_2}{P_1N_1}=\frac{P_1P_2}{OP_1}=\theta,$$

where θ is the angle P_1OP_2, measured in radians;

$$\therefore \ \text{time}=\sqrt{\frac{m}{\mu}}\cdot\theta.$$

Hence period $=2\pi\sqrt{\dfrac{m}{\mu}}$, and is independent of the amplitude. For the pendulum, of course $\mu=\dfrac{mg}{l}$, leading to the usual formula, $T=2\pi\sqrt{\dfrac{l}{g}}$.

It is also worthy of note (but not at all surprising) that here the motion of P is none other than that usually taken as the starting-point, uniform circular motion.

W. G. BICKLEY, B.Sc.

544. [A. 3.] $f(x)=ax^2+2hx+b. \quad \phi(x)=a'x^2+2h'x+b'.$

λ is any real quantity.

x_1, x_2 are the roots of the equation $f(x)-\lambda\phi(x)=0$, or

$$(a-\lambda a')x^2+2x(h-\lambda h')+b-\lambda b'=0. \quad \text{.......................(i)}$$

Now, if we put $\qquad y=ax^2+2hx+b, \quad$ ....(ii)

and eliminate x between (i) and (ii), the roots of the resulting quadratic in y are $f(x_1)\equiv ax_1^2+2hx_1+b$ and $f(x_2)=ax_2^2+2hx_2+b$.

From (i) and (ii),
$$ax^2+2hx+b-y=0,$$
$$a'x^2+2h'x+b'-\frac{y}{\lambda}=0,$$

whence
$$\frac{x^2}{hb'-\frac{hy}{\lambda}-h'b+h'y}=\frac{2x}{a'b-a'y-ab'+\frac{ay}{\lambda}}=\frac{1}{ah'-a'h}$$

or
$$\left\{y\left(\frac{a}{\lambda}-a'\right)-(ab'-a'b)\right\}^2=4(ah-a'h)\left\{hb'-h'b-y\left(\frac{h}{\lambda}-h'\right)\right\}.$$

Then $f(x_1)f(x_2)=$ product of the two values of y

$$=\frac{1}{\left(\frac{a}{\lambda}-a'\right)^2}\{(ab'-a'b)^2-4(ah-a'h)(hb'-h'b)\}$$

$$=\frac{\lambda^2}{(a-\lambda a')^2}\{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}.$$

This somewhat remarkable result shows that if for one set of values of λ, x_1, x_2, $f(x_1)f(x_2)$ is negative—then it is also true for any other set; and if for any one set of values of λ, x_1, x_2, $f(x_1)f(x_2)$ is positive—then it is also true for any other set.

The criterion $(ab'-a'b)^2 <$ or $> 4(ah'-a'h)(hb'-h'b)$ is precisely the condition that λ should be unrestricted or restricted in value. See Wolstenholme, Q. 217, p. 34.

R. F. Davis.

545. [D. 6. b.] *A Trigonometrical Lucubration.*

In his admirable article on Statical Stability in the May number, Dr. Brodetsky applies a trigonometrical theorem, the proof of which he leaves to the reader. This theorem, apart from its application, seems to be of interest.

The theorem is, that if $n>1$, $\dfrac{\sin n\theta}{\sin\theta}$ decreases as θ increases from zero. As Dr. Brodetsky remarks, it can be proved by differentiation; in fact, since $n\tan\theta-\tan n\theta$ is a factor of its derivative, the other factors being positive, the required result follows if we can assume that $n\tan\theta<\tan n\theta$, which is pretty obviously true, though a tyro might have a little difficulty in proving it analytically. It occurred to me to search for a more elementary proof. My first attempt resulted in a theorem which asserts less than that cited by Dr. Brodetsky, but is easily proved, and sufficient for the purpose.

In fact,
$$\frac{\sin na}{\sin a}-\frac{\sin 2na}{\sin 2a}=\frac{2\sin a\sin na(\cos a-\cos na)}{\sin a\sin 2a}>0,$$

if the angles are in the first quadrant, and $n>1$.

Hence
$$\frac{\sin n\frac{\theta}{2}}{\sin\frac{\theta}{2}}>\frac{\sin n\theta}{\sin\theta},$$

i.e. by halving the θ, we get a greater value of $\dfrac{\sin n\theta}{\sin\theta}$, and if this be done repeatedly, we see that $\dfrac{\sin n\theta}{\theta}$ is less than the limit it approaches for $\theta\to 0$.

In order to obtain a purely trigonometrical proof of the more general theorem as to the *continual* decrease of $\dfrac{\sin n\theta}{\theta}$, I tried to find an elementary proof that

$$\frac{\sin x\alpha}{\sin y\alpha} < \frac{\sin x\beta}{\sin y\beta}, \quad \text{where } x > y > 0 \text{ and } \alpha > \beta > 0,$$

by proving that $\sin y\alpha \sin x\beta - \sin x\alpha \sin y\beta$ is necessarily positive, under the conditions stated, and the restriction that the angles involved should be in the first quadrant.

The corresponding inequality for *cosines* is easily proved. In fact, we have

$$\begin{aligned} &2(\cos y\alpha \cos x\beta - \cos x\alpha \cos y\beta) \\ &\quad = \cos(y\alpha + x\beta) + \cos(y\alpha - x\beta) - \cos(x\alpha + y\beta) - \cos(x\alpha - y\beta), \end{aligned}$$

which is clearly > 0, since $y\alpha + x\beta < x\alpha + y\beta$,

and $\qquad \cos(y\alpha + x\beta) > \cos(x\alpha + y\beta)$, also $\quad y\alpha - x\beta < x\alpha - y\beta$,

whence $\qquad \cos(y\alpha - x\beta) > \cos(x\alpha - y\beta)$.

But the *sine* inequality I found much less tractable, by purely trigonometric analysis. Perhaps some one of your readers who may be interested might be more successful; but I became convinced that its proof, on lines similar to that above indicated, is essentially quite of a different order of difficulty, and I succeeded in getting only a manifestly positive form for

$$\sin y\alpha \sin x\beta - \sin x\alpha \sin y\beta$$

by a different and rather peculiar mode of expansion, resulting in an identity which I give here without proof.

I write $2m\theta$, $2n\theta$, $2m'\theta$, $2n'\theta$ for $y\alpha$, $x\beta$, $y\beta$, and $x\alpha$ respectively, m, n, m', n' being positive integers so that $n'm' = nm$ and n' is the greatest and m' the least of the four integers. To simplify the statement, I assume further that $n > m$; and I use $[r]$ as a contraction for $\cos r\theta$.

$$\begin{aligned} &\{\sin 2m\theta \sin 2n\theta - \sin 2m'\theta \sin 2n'\theta\} \div (2\sin^2\theta) \\ &\quad = (m - m')[0] + 2(m - m')\{[2] + [4] + [6] + \ldots + [2n - 2m]\} \\ &\qquad + (2m - 2m' - 1)[2n - 2m + 2] + (2m - 2m' - 2)[2n - 2m + 4] + \ldots \\ &\qquad + 1\,.\,[2n + 2m - 4m' - 2] + 0\,.\,[2n + 2m - 4m'] - [2n + 2m - 4m' + 2] \\ &\qquad - 2[2n + 2m - 4m' + 4] - \ldots - (m' + n' - m - n)[2n' - 2m'] \\ &\qquad - (m' + n' - m - n)\{[2n' - 2m' + 2] + [2n' - 2m' + 4] + \ldots + [2m + 2n]\} \\ &\qquad - (m' + n' - m - n - 1)[2m + 2n + 2] - (m' + n' - m - n - 2)[2m + 2n + 4] \\ &\qquad - \ldots - 2\,.\,[2m' + 2n' - 4] - 1\,.\,[2m' + 2n' - 2]. \end{aligned}$$

Here, in the right-hand member of the identity, the number of cosines having the sign $-$ prefixed is $2(m - m')(n - m')$, and is equal to the number of cosines having the sign $+$; and the angles of the former are all greater than the greatest angle belonging to the latter, so that their cosines have numerically smaller values. Thus this member, and therefore also the left-hand member is positive.

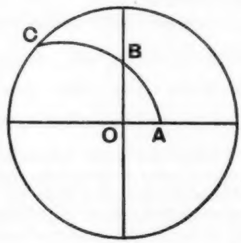
If it should happen that $m > n$, we have merely to interchange these two letters in the above formula, and the identity so changed is true for that case.

As this result is of rather a novel type among trigonometric formulae, I have thought it might be worthy of record. Of course it is rather too complicated to serve Dr. Brodetsky's didactic purpose.

It may be noted that the associated inequality $\tan x\alpha \tan y\beta > \tan y\alpha \tan x\beta$, where $(\alpha - \beta)(x - y) > 0$, shows that $\dfrac{\tan n\theta}{\theta}$ *increases* when θ increases from 0, if $n > 1$.

The three cognate inequalities involving *sines*, *cosines* and *tangents* respectively, in virtue of the identity $\sin\theta = \tan\theta\cos\theta$, are logically related inasmuch as the *cosine* inequality can at once be deduced from the other two; but as they are harder to prove, this relation is not of much importance. R. F. MUIRHEAD.

546. [U.] When paying a visit to the Pearl Mosque in the Fort at Agra in December, 1915, I noticed the sundial standing in the S.E. corner of the courtyard. The dial face is as the figure. OA is about $\frac{1}{3}$ of the radius,



OB is about $\frac{1}{2}$ of the radius, ABC is an arc. The diameters are at right angles. The gnomon is not there. Can somebody inform me how this worked? T. V. PHILPOTT.

547. [L[1]. 17.] *Intersections of Curves in Polars.*

An elementary point that is often overlooked is that the intersections of (say) $r=f(\theta)$ and $r=\psi(\theta)$ are not, *in general*, completely given by the equation

$$f(\theta)=\psi(\theta). \qquad \text{...(1)}$$

For example, the conics $\dfrac{10}{r}=2+5\cos\theta$ and $\dfrac{10}{r}=5-4\cos\theta$ intersect actually in four real points, two only of which are got by equating the values of r.

The root of the matter, of course, lies in the fact that the representation of the position of a point by polar coordinates is not unique, (r, θ) and $(-r, (2\kappa+1)\pi+\theta)$, ($\kappa$ integral), being identical.

In tracing polar curves, for some branches r is positive and for others negative. (1) gives only the vectorial angles of points of intersection of branches for which r has the same length. To get those of branches for which r has opposite signs, write one of the equations (say $r=\psi(\theta)$) in its equivalent form,

$$-r=\psi((2\kappa+1)\pi+\theta).$$

In this form the positive and negative branches exchange rôles, and we can now equate radii vectores again and get for the remaining intersections

$$f(\theta)+\psi((2\kappa+1)\pi+\theta)=0. \qquad \text{.................................(2)}$$

(1) and (2) together give the complete intersections of the curves.

Of course it often happens that the solutions of (2) are included in (1), but this is not true *in general*. E. H. SMART.

SOME INCIDENTAL WRITINGS BY DE MORGAN.

XV.

(*Continued from* p. 178, vol. ix.)

"The constant reference to this barleycorn measure (which is seldom, if ever, omitted) induced me to try what it really would make. There is some difference between the breadths of barleycorns. A certain statement of Thevenot (cited in the *History of Astronomy, Lib. Usef. Kn.*) makes the breadths of 144 grains of oriental barley give $1\frac{1}{2}$ French feet; at which rate 64 would only give 8 53 inches English. A sample from a London shop gave me (when the largest grains were picked out) 33 to just more than five inches. Some other samples, procured from two different parts of England as the finest which could be got, gave 33 to 5 inches, 33 to 5·1 inches, and 33 to 5·1 inches. The average of these is 9·8 inches to 64 grain-breadths: a result which coincides more nearly than could have been expected with the following determinations.

There is a chain of writers who have studied to perpetuate their geometrical foot by causing a line to be laid down on the page, representing the digit, the palm, or the foot. Sometimes the palm or foot is divided into digits: and of course I rely most on those whose subdivisions are the best. As paper is apt to shrink as it becomes old, the foot deduced from these will be somewhat too small, and it may be afterwards discussed how much it should be lengthened. Leaving this for the present, I give the measurements from different authors.

"*Margarita Philosophica*, above described. In the Strasburg edition of fifteen-four, the length of the geometrical palm is less than $2\frac{1}{2}$ inches by from half to three-fourths of the 24th of an inch. Taking it half-way between these, the four-palm foot is 9·9 inches English. In the Basle edition of fifteen-eight, in which the woodcuts are of much rougher execution, the palm is 2·64 inches, giving a foot of 10·56 inches. The palm only is given in both cases.

"*Oppenheim*, fifteen-twenty-four. Stöffler, Elucidatio Fabricae Ususque Astrolabii. *Folio in threes* (quarto size).

The digit, palm, and foot are separately given; the foot is divided into palms, and all agree excellently well with one another. The foot is exactly 9·75 inches English.

[Fernel, Jean (1497-1558). Born at Clermont; studied mathematics and astronomy at the College of Sainte-Barbe, Paris. His works on these subjects were: *Monalosphaerium, sive astrolabii genus, generalis horarii structura et usus*, 1526; *Cosmotheoria*, 1528; *De proportionibus*, 1528. He then took to medicine; was notable for his revival of interest in the study of the Greek physicians; wrote several medical works; was appointed Court physician, and left a large fortune. The *Cosmotheoria* gives an account of his determination of a degree of the meridian, made by counting the number of revolutions in his carriage wheels between Paris and Amiens. His astronomical observations were made with a triangle used as a quadrant, and his resulting length of a degree was very near the truth. *Encyc. Brit.* "Fernel," x. p. 281; "Figure of the Earth," viii. p. 801.]

"*Paris*, fifteen-twenty-six. John Fernel. "Monalosphaerium." *Folio in threes.*

"*Paris*, fifteen-twenty-eight. John Fernel.* "Cosmotheoria." *Folio in threes.*

"The historical mistake arising out of these works is the most remarkable circumstance attending the loss of the geometrical foot. While Fernel was publishing the first work, he was meditating (or perhaps executing) his famous measurement of a degree of the meridian. In this first work he lays his geometrical foot down the page, with great care, as he says (*omni molimine*). In two copies of this work which I have examined, the length of the foot is

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* Of Fernel's *De proportionibus Libri duo*, 1528, De Morgan says: "A book on proportion of this date is in great part filled with Boethian arithmetic. This book deserves attention: its author had a much better grasp of Euclid than most of his contemporaries."

within a sixtieth of an inch of nine inches and two-thirds, giving 9·65 inches. In the second work, in which he announces his measure of the degree, he states that five of his paces and those of men of ordinary stature make six *geometrical paces*; which, he adds, is agreeable to the opinion of Campanus and others (at least a century and a half before), who made the mile of 1000 common paces to be 1200 geometrical paces. Allowing 60 inches (English) to a common pace, which is rather over than under the truth, this gives a geometrical pace of 50 inches, and a foot of *ten* inches. This is enough to show that Fernel was, in the second instance, speaking in rough terms of the foot which he printed *omni molimine* in the first. The geometrical pace being forgotten, and the *monalosphaerium* also, the modern historians have assumed that Fernel used the Paris foot: by which he is made to come very near the real degree, whereas he is fifteen miles wrong."

"*Paris*, fifteen-fifty-two. Jac. Koebelius. "Astrolabii Declaratio." *Octavo.*

"This worthy astrologer, after referring to the perfect notoriety of the system of measures, gives a digit and a palm. The digit is nine-sixteenths of an inch (English), giving a foot of nine inches. The palm is 2½ inches and one-sixteenth, giving 10 inches and a quarter to the foot. The book is small, and the palm incorrectly subdivided."

[Köbel, Jacob. ASTROLABII DECLARATIO, *ejusdem Usus mire jucundus, non modo Astrologis, Medicis, Geographis, etc. utilis ac necessarius: verùm etiam Mechanicis quibusdam Opificibus non parum commodus; facilioribus Formulis nuper aucta, longéque evidentior edita; accessit* ISAGOGON *in* ASTROLOGIAM JUDICIARIUM, 16mo, pp. 65, Coloniae, 1594. Contains a chapter on mathematical instruments and their use (Sotheran). He is described by Poggendorf as "Stadtschreiber in Oppenheim," where he died in 1533. He also wrote *Eyn New geordnet Vysirbuch*, etc., 4to, Oppenheim, 1515; *Rechenbuch*, etc., 4to, Augsberg, 1514; *Sonnen-Uhr vom Schatten des Menschen, item Sonnenuhr von d. linken Hand.*, 8vo, Maynz, 1534; *Geometrey*, etc., 1598, 1618. But of the latter, Sotheran had lately a much earlier edition:

GEOMETREI. *Von* KÜNSTLICHEM FELDMESSEN *vnnd* ABSEHEN | *Allerhand Höhe | Fleche | Ebne | Weitte vnnd Breyte: Als Thürn | Kirchen | Bäw | Bäum | Felder vnd Echer ꝛc. Mit fast wercklich vnd künstlich zubereytem Jacob Stab | Philosophischen Spiegel | Schatten | vñ Messruten . . . Dabei | von bereytung | verstand vnd vilfaltigem nützlichen* GEBRAUCH *des* QUADRANTEN, sm. 4to, Franckfurt am Meyn, Christian Engenolffs Erben, 1563.

Sotheran (hereinafter S.) quotes from the Libri Catalogue: This work on land-surveying and the use of the quadrant is far rarer than his Arithmetic, or his *Astrolabii Declaratio*. In it he makes use of Arabic numerals; and adds that the finer wood-engravings by Jost Amman add greatly to the value of the work.

The *Eyn New geordnet Vysirbuch* is described, p. 11, *Arithmetical Books*, as "A work on gauging, in which the Arabic numerals are used." De Morgan, *loc. cit.* p. 10, gives also:

Augsberg, fifteen-fourteen. Jacob. Kobel (printer). *Ain Nerv geordnet Rechen biechlin auf den linien mit Rechen pfeningen*: etc., 12mo size.

"Computation by counters and Roman numerals: the Arabic numerals are explained but not used. In the frontispiece is a cut representing the mistress settling accounts with her maid-servant by an abacus with counters. This book is said by Kloss to have been also printed by Kobel himself at Oppenheym in the same year." De Morgan, *loc. cit.* p. 10.

Peter Ryff (1552-1629), born and died at Basel, where he was a physician in practice, and later Professor of Mathematics at the University.

Ryff, Peter. *Quaestiones Geometricae in Euclidis et P. Rami Στοιχείωσιν quibus* GEODAESIAM *adjecimus per usum Radii Geometrici*, 4to, Francofurti, 1621. *Editio Altera, cui accessii* COMMENTATIO OPTICA, *sive brevis Tractatio de Perspectiva Communi*, 12mo, Oxoniae, 1665.

Poggendorf gives 1600 as the date of the *Quaestiones Geometricae in Euclidis Elementa*, 1606 for the *Quaestiones . . . cum geodaesia per usum radii geometrici prodierunt*. He also mentions: *Compendium arithmeticae Vrstisii*, 8vo, Oxoniae, 1626; *Elementa sphaerae mundi*, 8vo, Oxoniae, 1627; and *Ephemerides*, with astrological prognostications for a series of years.]

Of the *Quaestiones* (1621) De Morgan says that: "Four very accordant palms are given, indifferently subdivided into digits. Each palm is 2¼ inches and three-sixteenths, giving a foot of 9·75 English inches."

He then sums up: "From these different sources, good and bad, we have for the geometrical foot 9·8, 9·9, 10·56, 9·75, 9·65, 10, 9, 10·25, 9·75 inches: the mean is 9·85. But much the best authorities are Fernel and Stöffler, because they are the greatest names, have given the whole foot, and have taken the greatest pains with the subdivisions. Their results are 9·75 and 9·65, with a mean of 9·7."

"Taking this as the foot on paper, it remains to ask how much it must be lengthened to allow for the shrinking of the paper. At first, relying on the plate in Dr. Bernard's work on ancient weights and measures, in which the English foot appears to have shrunk by its 42nd part, I was disposed to lengthen the above in the ratio of 41 to 42. But observing that an older English foot, figured in the *Pathway to Knowledge*, 1596, has shrunk only by its sixtieth part, I am rather inclined to consider the shrinking of Bernard's as an extreme case. And moreover, the two copies of the *Monalosphaerium* give the same foot within one-hundredth of an inch certainly, and less: and it is very unlikely that if the paper had shrunk much, it should have shrunk so equally in two different copies. But, taking one-fiftieth as the outside, it follows that the geometrical foot is anything the reader pleases between 9·7 and 9·9 English inches. The result from modern barley * gives 9·8, as above shown.

It is remarkable how completely the English writers are in ignorance of the existence and use of the geometrical foot among their continental neighbours. Blundeville takes Stöffler, in the work above mentioned, to be speaking of a *German* foot, which, says he, 'Stöffler makes to be 2½ inches less than the English foot.'"

[S. recently had a copy of Ramus, Petrus (Pierre de la Ramée), ARITHMETICES *Libri II., et* ALGEBRAE *totidem:* a LAZARO SCHONERO emendati et explicati: ejusdem SCHONERI *Libri II*.: DE NUMERIS FIGURATIS: DE LOGISTICA SEXAGENARIA, 12mo, Francofurdi, 1586; SCHOLARUM METAPHYSICARUM *Libri XIV*., in totidem Metaphysicos Libros Aristotelis, 12mo, Parisiis, 1566; SCHOLARUM PHYSICARŪ *Libri VIII*., in totidem acroamaticos libros Aristotelis, 12mo, 1565; SCHOLARUM MATHEMATICARUM, *Libri XXXI*., 4to, Basileae, Eus. Episcopius, 1569.

S. reminds us that Schonerus perished in the massacre of St. Bartholomew.

The copy of the *Arithmetices* which De Morgan examined is dated 1592. He states that the first edition is said to be "of fifteen-eighty-four."

Ramus, VIA REGIA AD GEOMETRIAM. THE WAY TO GEOMETRY, being necessary and usefull, for Astronomers, Geographers, Land-meaters, Sea-men, Engineres, Architecks, Carpenters, Paynters, Carvers, etc., now translated and enlarged by WILLIAM BEDWELL, sm. 4to. Printed by Thomas Cotes, 1636.

Bedwell was vicar of St. Ethelburga's, Bishopsgate St., one of the translators of the Authorised Version, and the Father of Arabic studies in England. The above volume is posthumous. He was a friend of Briggs and Greaves. He had previously translated the "*De Numericis Geometricis*. Of the Nature and properties of geometrical numbers, first written by Lazarus Schonerus, and now Englished, 4to, London, 1614"; treating "of figurate, square, etc. numbers, with applications to mensuration."

The name "de la Ramée" survives—at any rate in the Channel Islands. It was the maiden name of "Ouida."

Stöffler, Johann. Univ. Tübingen. CALENDARIUM ROMANUM MAGNUM (in fine: ABACUS REGIONUM, *Principatuū, Ducatuū, Satrapiarū, Marchiarū Comitatuum, Prouinciarū, Insularū, Peninsularu, et Oppidorum nobiliorū, aut cognobiliorū per totā ferme Europā*. Oppenheym per Jacobum Köbel, 1518.

ELUCIDATIO FABRICAE USUSQUE ASTROLABII, *cui multa et diligens accessit Recognitio, unà cum Schematum Negotio accomodatorum, exactissima Expres-*

* "Those who have tried to make the *lengths* of three barleycorns into an inch will probably think little of this mode of judging. But I observed that in samples of very different apparent fineness, the difference was in the length of the corns, the breadths hardly varying at all."

sione, et Indice Rerum et Verborum copiosissimo, Lutetiae, Gul. Cauellat, 1553, 12mo.

These are both given by S. The latter is said to be printed in italics. The former is described by De Morgan in the account of Stöffler mentioned below.

In PROCLI DIADOCHI, *Authoris grauissimi* SPHAERAM MUNDI, *omnibus Numeris longè absolutissimus* COMMENTARIUS, *ante hac nunquam Typis excusus*, fol. Tubingae, ex offic. Hulderrichi Morhart, 1534.

COELESTIUM RERUM DISCIPLINAE, *atque totius* SPHAERICAE, *variorum* ASTROLABIORUM COMPOSITIONEM *seu* FABRICAM, *necnon eorundem Usuum ac variarum Utilitatum Explanationem . . . in aliquot Locis praecipuè cum Propositionibus, tum earum Expositionibus, in meliorem Formam redigenda, atque imprimenda curavimus*, Moguntiae, Petrus Jordan, 1535.

S. also gives a rare edition of the *Ephemeridum Opus*, of which he quotes from the Libri Catalogue: "The rarity of these Ephemerides is such that Delambre was unable to see a copy."

EPHEMERIDUM OPUS, *a capite anni Redemptoris Christi* 1532 *in alios* 20 *proxime subsequentes: ad veterum imitationē accuratissimo calculo elaboratum.* Gothic letter, Venetiis, Petrus Liechtenstein, 1532.

An account is given of John Stöffler on pp. 29-31 of De Morgan's essay "On the Earliest Printed Almanacs," which opened the *Companion to the Almanac* for 1846. It contains, pp. 27-8, an interesting investigation of a so-called prophecy connected with Stöffler.]

XVI.

[A correspondent had asked where the paradox of Achilles and the Tortoise is to be found.]

I. ii. 186. **Achilles and the Tortoise.**—Your correspondent will find references in the article "Zeno (of Elea)" in the *Penny Cyclopaedia.* For Gregory St. Vincent's treatment of the problem, see his *Quadratura Circuli,* Antwerp, 1647, folio, p. 101, or let it alone. I suspect that the second is the better reference. Zeno's paradox is best stated, without either Achilles or tortoise, as follows: No one can go a mile; for he must go over the first half, then over half the remaining half, then over half the remaining quarter; and so on *for ever.* Many books of logic, and many of algebra, give the answer to those who cannot find it. M.

[I do not think that the article on Zeno alluded to was written by De Morgan. It is not attributed to him in the (incomplete) list given in his wife's *Memoir.*]

I. ii. 186. ACHILLES AND THE TORTOISE.—This paradox, whilst one of the oldest on record (being attributed by Aristotle to Zeus Eleates, B.C. 500) is one of the most perplexing, upon first presentation to the mind, that can be selected from the most ample list. Its professed object was to disprove the phenomenon of motion; but its real one, to embarass an opponent. It has always attracted the attention of logicians; and even to them it has often proved embarassing enough. The difficulty does not lie in proving that the conclusion is absurd, but *in showing where the fallacy lies. . . .*

Whately's Logic, 9th edition, p. 373.—This is one of the sophistical puzzles noticed by Aldrich, but he is not happy in his attempt at a solution. He proposes to remove the difficulty by demonstrating that in a certain given time Achilles *would* overtake the tortoise; as if any one had ever doubted *that.* The very problem proposed, is to surmount the difficulty of a seeming demonstration of a thing palpably impossible: to show that *it is* palpably impossible is no solution of the problem.

I have heard the present example adduced as a proof that the pretensions of logic are futile, since (it was said) that the most perfect logical demonstration may lead from true premises to an absurd conclusion. The reverse is the truth; the example before us furnishes a confirmation of the utility of an

acquaintance with syllogistic form, *in which form the pretended demonstration in question cannot be exhibited.* An attempt to do so will evince the utter want of connection between the premises and the conclusion.

What the Archbishop says is true, and it disposes of the question as one of "Formal Logic": but yet the form of the sophism is so plausible that it imposes with equal force on the "common sense" of all those who repose their conclusions upon the operations of that faculty. With them a different procedure is necessary; and I suspect that if any of the most obstinate advocates of the sufficiency of common sense for the "balancing of evidence" were to attempt the explanation of a hundred fallacies that could be presented to him, he would be compelled to admit that a more powerful and a more accurate machine would be of advantage to him in accomplishing his task. This machine the syllogism supplies.

The discussion of Gregory St. Vincent will be found at pp. 101-3 of his *Opus Geometricum*, Antw. 1647, fol. The principle is the same as that which Aldrich afterwards gave. I can only speak from memory of the discussion of Leibnitz, not having his works at hand; but I am clear in this, that his principle again is the same. . . . St. Vincent . . . indeed uses lines to represent the spaces passed over; and their discussion occurs in a chapter on what is universally (but very absurdly) called "geometrical proportion." It is yet no more *geometrical* than our school-day problem of the basket and the hundred eggs in Francis Walkinghame. Mere names do not bestow character, however much *philosophers as well as legislators* may think so. All attempts of the kind have been, and must be, purely numerical. T. S. D.

[Thomas Stephens Davies (1795-1851), Mathematical Master, R.M.A., Woolwich, 1834.]

[Gregorius a Sto. Vincentio, S.J. *Opus Geometricum de Quadratura Circuli et Sectionum Coni X Libris comprehensum*, roy. fo. Antverpiae, 1647-9.

Opus Geometricum posthumum ad Mesolabium per Rationum Proportionalium Novas Proprietates. Finem Operis Mors Authoris antevertit, large fo. Gandavi, 1668.

On this ancient paradox we may refer to the papers in the *American Mathematical Monthly*, 1915, on *The History of Zeno's Arguments on Motion*, by Prof. Cajori; to Bertrand Russell's *Principles of Mathematics*, i. pp. 346-354, and to P. E. B. Jourdain's shrewd and witty essay on the kindred paradox of *The Arrow* in *The Monist*.]

XVII.

I. ii. 193. **Notes and Queries.**—The history of books and periodicals of a similar character ought to be an object of interest to the readers of this work. The number of works in which answers have been given to proposed questions is not small. Not to mention the *Spectator* and its imitators, nor the class of almanacs which give riddles and problems, nor mathematical periodicals of a more extensive character,—though all these ought to be discussed in course of time,—there yet remains a class of books in which general questions proposed by the public are answered periodically, either by the public or by the editors. Perhaps an account of one of these may bring out others.

In 1736 and 1737 appeared the *Weekly Oracle*; or, *Universal Library. Published by a Society of Gentlemen.* One folio sheet was published weekly, usually ending in the middle of a sentence. (Query. What is the technical name for this mode of publication? If none, what ought to be?) I have one folio volume of seventy numbers, at the end of which notice of suspension is given, with prospect of revival in another form: probably no more was published. The introduction

is an account of the editorial staff : to wit, a learned divine who "hath entered with so much discernment into the true spirit of the schoolmen, especially Thomas Aquinas and Duns Scotus, that he is qualified to resolve, to a hair's breadth, the nicest cases of conscience." A physician who "knows, to a mathematical point, the just tone and harmony of the rising pulses. . . ." A lawyer who "what he this day has proved to be a contingent remainder, to-morrow he will with equal learning show must operate as an executory devise or as a springing use." A philosopher "able to give the true reason of all things, from the composition of watches, to the raising of minced pies . . . and who, if he is closely questioned about the manner of squaring the circle, or by what means the perpetual motion, or longitude, may be discovered, we believe has honesty, and we are sure that he has skill enough to say that he knows—nothing of the matter." A moral philosopher who has "discovered a *perpetuum mobile* of government." An eminent virtuoso who understands "what is the best pickle to preserve a rattle-snake or an Egyptian mummy, better than the nature of the government he lives under, or the economy and welfare of himself and family." Lastly, a *man of mode*. "Him the beaus and the ladies may consult in the affairs of love, dress, and equipage."

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47. *O.* I was thirty-three years of age * when I bought this excellent little library. I could hardly believe that I possessed such a treasure when I looked back on the day when I first saw the mysterious word "Algebra," and the long course of years in which I had persevered almost without hope. It taught me never to despair. I had now the means, and pursued my studies with increased assiduity; concealment was no longer possible, nor was it attempted. I was considered eccentric and foolish, and my conduct was highly disapproved of by many, especially by some members of my own family. . . .

. . . (Professor Playfair) knew that I was reading the *Mécanique Céleste* and asked me how I got on? I told him that I was stopped short by a difficulty now and then, but I persevered till I got over it. He said, "You would do better to read on for a few pages and return to it again, it will then no longer seem so difficult." I invariably followed his advice and with much success. . . .—*Memoirs of Mrs. Somerville*, pp. 80-81.

48. *P.* [In 1832, Poisson advised her to complete the *Mechanism of the Heavens*, by writing a volume on the form and rotation of the earth and the planets.]

. . . I now began the work, and, in consequence, I was led into a correspondence with Mr. Ivory, who had written on the subject, and also with Mr. Francis Baily, on the density and compression of the earth. My work was extensive, for it comprised the analytical attraction of spheroids, the form and rotation of the earth, the tides of the ocean and atmosphere, and small undulations.

When this was finished, I had nothing to do, and as I preferred analysis to all other subjects, I wrote a work of 246 pages on curves and surfaces of the second and higher orders. While writing this, *con amore*, a new edition of the *Physical Sciences* was much needed, so I put on high pressure and worked at both. Had these two manuscripts been published at that time, they might have been of use; I do not remember why they were laid aside, and forgotten till I found them years afterwards among my papers. Long after the time I am writing about, while at Naples, I amused myself by repairing the time-worn parts of these manuscripts, and was surprised to find that in my eighty-ninth year I still retained facility in the "Calculus."—*Loc. cit.*, p. 202.

* [1813].

REVIEWS.

Mathematics for Engineers. Part II. By W. N. Rose. Pp. xiv, 420. Price 13s. 6d. 1920. (Chapman & Hall.)

This volume forms the second part of a work belonging to the D.U. (directly useful) series founded by the late Wilfrid J. Lineham.

Chapters I.-XI. deal with the Calculus, XII. with the solution of spherical triangles, XIII. with Mathematical Probability and with Least Squares. Four-figure tables are added. These are more than usually complete, for they include Napierian logarithms, extended tables of Natural and Logarithmic sines, cosines and tangents, as well as Exponential and Hyperbolic Functions.

As with Part I., while almost all rules and processes of merely academic interest have been cut out, yet the importance and necessity of logical reasoning have been studiously maintained. Thus, the idea of the calculus is founded on algebraic processes; graphs are made a subordinate but necessary complement. As an instance of the careful presentation of the fundamental ideas, we may note the remark made by the author at the foot of p. 11, that the limiting value of the gradient of a curve *does not take the indeterminate form* $\frac{0}{0}$, but is given by a definite figure. On the other hand, it seems at first sight that "slope curves" (first derived curves) are introduced too early, on p. 13, and that this is inconsistent with the author's intention, as expressed above. Another small point—it strikes me as unwise at this stage (p. 26) even to make a passing mention of the *operator D.* My experience with engineering students has been that, up to a certain point, the longer one can conceal from the student that he is doing anything but ordinary algebra, the better it is for him. There seems to be some psychological effect attached to the words "calculus" and "operator" that conjures up a mental image of a "Col. Bogey," whom it is impossible to beat; and this induces an unconscious passive resistance that is difficult to overcome. After the essential operations have been mastered, the ill-effect of the words disappears.

The fundamental formulae are developed with great care; but once these are obtained, say from p. 150 onwards, the further development seems to me to proceed far too rapidly. Thus, on p. 163, but fifty pages from the first paragraph on the meaning of integration, we have the evaluation of the difficult integral, $\int_0^\infty e^{-x^2}dx$, involving a change in the order of integration. With reference to the fundamental formulae, I am inclined to think that the standard processes adopted might have been replaced by more modern methods. Thus, instead of the differentiation, by the use of the Binomial Theorem, of a power of the *independent* variable, I should like to see the differentiation of a product of two *dependent* variables made the fundamental theorem, especially in a book in which graphs are subordinate. This avoids a difficulty, which is a *pons asinorum* to almost all beginners; namely, the introduction of the factor dy/dx when he proceeds from the differentiation of x^n to that of y^n, where y is a function of x. Also I think that the introduction of the summarised rule for products and quotients, viz.,

$$\frac{d}{dx}\cdot\frac{uv}{wz}=\frac{uv}{wz}\left(\frac{1}{u}\cdot\frac{du}{dx}+\frac{1}{v}\cdot\frac{dv}{dx}-\frac{1}{w}\cdot\frac{dw}{dx}-\frac{1}{z}\cdot\frac{dz}{dx}\right),$$

would have been a useful addition. For the differentiation of sines, cosines and tangents, it seems a pity that the extremely simple geometrical proofs are not given, at any rate as alternatives to the trigonometrical proofs depending on the limiting value of $\sin x/x$; if only for the reason that these geometrical proofs bring out the fact more clearly that the angle is measured in radians.

The great feature of the book is the happy manner, as befits the name applied to the series to which it belongs, in which each fresh instalment of theory is applied immediately to some useful and interesting practical example. Nothing but praise can be given to the applications of the Integral Calculus in Chap. VII. The examples are full of interest; without doubt, this and Chap. X.

are the best chapters in a book full of good things—a book which, while it is hardly suitable for a beginner working privately, is one that, if used under the guidance of an experienced teacher, can hardly be matched.

With regard to notation, the author uses i for steady current, which apparently forces him on p. 282 to utilise j for the imaginary unit. It seems a pity that he is not content to use the Greek letter iota, which is now fairly standardised. Also, instead of the large L for limit, one would prefer to see the usual lim.

I noticed only one misprint, in l. 5 up, p. 165. J. M. CHILD.

Leçons sur l'approximation des Fonctions d'une Variable Réelle. By C. DE LA VALLÉE POUSSIN. Pp. 150 + vi. 12 fr. 1919. (Gauthier-Villars.)

M. S. Bernstein, in a paper, "Sur les Recherches Récentes relatives à la meilleure approximation des fonctions continues par des polynômes," read at the Fifth International Congress of Mathematicians in 1912, very clearly stated the problem dealt with in this book. Since Weierstrass, in 1885, enunciated his classic theorem "that any function, continuous in an interval (a, b), can be developed in a series of polynomials, uniformly convergent in that interval," various mathematicians have given different proofs of the theorem and have constructed different polynomials of degree n, $R_n(x)$, such that the maximum value of the difference $|f(x) - R_n(x)|$ in the interval tends to zero, as n increases indefinitely. The approximations obtained by different methods, for the same function and for the same value of n, are not always the same, and this fact has led to the search for those polynomials $P_n(x)$ for which the maximum value of the difference $|f(x) - P_n(x)|$ tends most rapidly towards zero. The polynomials, $P_n(x)$, which have this property, are called "Polynomials of Approximation," and the minimum value, $E_n[(x)]$, of the modulus $|f(x) - P_n(x)|$ is called "the best approximation" of the given function in the interval. Polynomials of approximation had been used even before the discovery of Weierstrass, but it is only during the last twelve years that the systematic study of the magnitude of the best approximation, $E_n[f(x)]$, has been undertaken. Yet a consideration of this magnitude is necessary to complete Weierstrass' Theorem, which really implies that $\underset{n\to\infty}{\mathrm{Lt}}\, E_n[f(x)] = 0$, whatever continuous function of x, $f(x)$ may be.

The author of this book was (in 1908) among the first to consider the problem systematically and he was followed by Dunham Jackson (1911) and S. Bernstein (1911 and 1912). Their results are summarised by M. Bernstein in the paper mentioned at the beginning of this review. Jackson and Bernstein published further papers in 1913. It is probable that the war delayed further research on the subject by our author, who is Professor of Mathematics at ill-fated Louvain. However, during the dark days, he received hospitality from Paris, and the early months of 1918 brought new and important results from his pen. The present volume is, in the main, the reproduction of a series of lectures delivered at the Sorbonne during May and June of that year.

The main fact that emerges from the various researches is the existence of a close connection between the differential properties of the function $f(x)$ and the asymptotic law of the decrement of $E_n[f(x)]$. This book treats of this reciprocal relation. It shows that, whether the development is by polynomials or by trigonometric expressions, the order of the best possible approximation depends on the continuity and differential properties of the function or, in the case of an analytic function, on the nature and situation of the singular points. Conversely, if it is given that a function can be represented with an approximation of a certain order, it is shown that the differential properties of the function can be deduced. The author has, of course, profited largely by the work of Bernstein and Jackson, but he has made the subject peculiarly his own and he has welded the various results into one synthetic whole. In many cases, he has extended Bernstein's results and, at times, has replaced Bernstein's proofs by others, more accurate or more elegant. Jackson obtained results first for the representation by polynomials and deduced from these corresponding theorems for the trigonometric representation. M. de la Vallée Poussin very skilfully reverses this order; that his is the more natural order is evident from the greater power and the more general results obtained by his treatment.

The last two chapters, on "Fonctions analytiques présentant des certaines singularités," well illustrate the power and elegance of his method.

Even if M. de la Vallée Poussin's name were not on the title page, the inclusion of this volume in the remarkable "Collection de Monographies sur la Théorie des Fonctions," edited by M. Émile Borel, would be a sufficient guarantee of its authority and general excellence. A. DAKIN.

Elements of Vector Algebra. By L. SILBERSTEIN. Pp. iv + 42. 5s. net. 1919. (Longmans.)

This little book was written on the suggestion of Messrs. Adam Hilger, and contains the work required for reading the *Simplified Method of Tracing Rays*, by the same author. It gives an account of the addition of vectors, of their scalar multiplication (indicated by juxtaposition) and of their vector multiplication (indicated by a prefixed V). The fundamental formulae are deduced and some geometrical applications given. This fills just more than half the book. Then follows a short account of linear vector operators, of dyads and dyadics, and some hints on differentiation. A few illustrations would have been helpful here. With this book alone, the reader who is ignorant of, say, Hydrodynamics or Elasticity, would be at a loss to understand what applications could be made of the latter part. H. G. F.

Fermat's Last Theorem. (Revised Edition.) By M. CASHMORE. Pp. 55. 2s. 1918. (G. Bell & Sons.)

Three attempted proofs are given of this well-known puzzle. Unfortunately all are fallacious. H. G. F.

Les spectres Numériques. By M. PETROVITCH. With preface by M. ÉMILE BOREL. 1919. (Gauthier-Villars.)

The defect of this book is that there is nothing in it. It may seem very improbable that a book published by Gauthier-Villars, and introduced by M. Borel, should contain no proposition of interest. All that I can say is that I can find none, and that, reading between the lines of M. Borel's preface, I am inclined to suspect that that very eminent mathematician is of approximately the same opinion.

Given a series of integers, say

$$31, 17, 3, 169, 24, \ldots,$$

we can, in an infinity of ways, embody them, interspersed with zeros, in a sequence

$$31017000300169000024\ldots;$$

and such a sequence may be called a "numerical spectrum". Again, given a function $f(z)$, there may be an operation Δ such that

$$\Delta f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

is a power series with integral coefficients. From these coefficients we may form a "spectrum", and we may, if we please, call this a "spectrum of f relative to Δ".

On these foundations M. Petrovitch builds an elaborate structure of definitions. These would be justified if some application of interest could be found for them. All that appears from the book is that M. Petrovitch has found none. G. H. HARDY.

The Theory of the Imaginary in Geometry. By J. L. S. HATTON. Pp. vi + 216. 18s. net. 1920. (Camb. Univ. Press.)

In writing this book Prof. Hatton had a great opportunity; for the subject is one of extreme importance both logically and didactically, and there is no English treatise* in which it is dealt with adequately. The book, moreover, shows abundant evidence of thought and ingenuity. If then I am disappointed with the result, it is because I differ fundamentally from the author in my judgment as to what methods of developing the theory are, in the light of modern knowledge, the best.

* No reader of Prof. Veblen's admirable *Projective Geometry* will suspect me of using "English" to include "American". I ought perhaps to except Mr. Mathews' *Projective Geometry*; but Mr. Mathews passes over these particular matters rather lightly.

Prof. Hatton's outlook is substantially that of von Staudt. The elegance of von Staudt's theory is beyond question, and it is even now of much historical importance. Von Staudt's point of view may be summarised shortly as follows. Real geometry having been established, or being taken for granted, imaginary geometry is to be grafted on to it. The process is essentially similar to that by which the theory of irrational number is grafted on to that of rational number in analysis. "Real" geometry is found to be honeycombed with exceptions and distinctions until it has become aesthetically intolerable. Homographies may or may not have double elements, straight lines may or may not meet conics; just as, in the arithmetic of rationals, $x^2=a$ may or may not have roots. And the process by which these intolerable anomalies are banished is simply that of replacing the rational number in one case, the "real point" in the other, by some wider and more complex logical construction. In the one case the *deus ex machina* is the *section* of Dedekind, in the other the *involution*.

We begin by replacing *pairs of real points* by *involutions*, the involutions of which they are the double points. These involutions are necessarily hyperbolic, and every proposition about two real points may be translated into the new language and restated as a proposition about a hyperbolic involution. But, as soon as this step has been taken, the restriction to *hyperbolic* involutions appears to be entirely artificial; most of our propositions may be stated in a form in which they are equally true when the involutions considered are elliptic. We obtain a geometry of involutions valid whether the involutions considered have double points or not; but its propositions correspond to the propositions of the older geometry in the one case and not in the other. The older language, however, is more natural and more suggestive, and in order to preserve it we agree to speak not of an elliptic involution but of a "pair of complex points". The pair of complex points *is*, by definition, the involution. "Complex point" is an *incomplete symbol*, to use Mr. Russell's illuminating phrase.

Prof. Hatton does not seem to me, in his opening sections, to state this point of view with all the clearness that is desirable. He begins with an "Axiom": "every overlapping involution determines a pair of points . . ." To this, I think, there is an obvious and unanswerable objection. These "points" are *ex hypothesi* not "real", for if they were the axiom would be false. They are "complex"; but "complex points" have not been defined, and therefore the axiom is meaningless. Prof. Hatton is, in short, trying to state as an *axiom* what can, from the nature of the case, be only a *definition*.

The point may seem abstract and "philosophical"; and certainly it is irrelevant to the later development of the theory. Its logical importance, however, is obvious, and its pedagogical importance, I should have thought, even more so. It is sure to puzzle a student; the more intelligent he is the more certain it is that he will be puzzled; and if he ultimately decides that Prof. Hatton's axiom is devoid of meaning, I think he will be entirely in the right.

In any case this is not my main point. My principal objection to Prof. Hatton's treatment is that von Staudt's procedure has long ceased to be a natural one, or one suitable for instruction of a comparatively elementary kind. It was natural only so long as it could be supposed that "real" space and "real" geometry were in some sense logically prior to, or more firmly founded than, "complex" space and geometry; and this is an exceedingly Victorian view. It is natural when we begin, and no doubt we must all pass through it; but an intelligent reader ought to have got beyond it long before he approaches the subject matter of Prof. Hatton's book. Any university student should be able to understand that the propositions of geometry are not affected by holes in a blackboard, or the presence or absence of Einstein's gravitational field; and, so soon as this is grasped, the supposed priority of "real" geometry, resting as it does on a mere naïve appeal to the reputed facts of physical experience, becomes ridiculous, and the adoption of some more abstract view imperative.

If this is granted, two courses are open to us. One is to follow Prof. Veblen, and to develop projective geometry boldly from the outset as an abstract science founded on a definite system of axioms; and this is no doubt the ideal at which we should aim. The supposed reference to real or physical space

then stands out clearly as an obvious bogey, and, so far from the complex geometry appearing as a graft upon the real, we find the real geometry appearing as a specialisation of the complex.

This is the ideal: it may be, under present conditions, a difficult one. There is a natural and much easier path, and that is to base the whole theory of the imaginary in geometry upon the ordinary methods of analysis. And in this, for my part, I subscribe to the opinion of Darboux. "Il y a là" (that is to say, in the older geometrical theories) "quelque chose d'artificiel; le développement de la théorie est nécessairement un peu compliqué. Aussi cette méthode de von Staudt n'a pu réussir à prévaloir. Il semble que, pour l'introduction et l'interprétation des imaginaires, il vaut mieux s'en tenir à la méthode analytique qui repose sur l'emploi des coordonnées rectilignes." To that I would assent whole-heartedly, with a reservation in favour of the study, later at all events, of the exceedingly beautiful axiomatic developments laid down by the modern projective geometers.

I have in any case no sympathy with attempts to treat everything, suitable or unsuitable, by synthetic methods. Elementary synthetic geometry contains a great deal that is very elegant and of very high educational value. But it has been made a fetish in English mathematical education, an excuse for mere triviality, and for ignorance of, or apathy towards, the great lines of advance of modern mathematics. Simple configurations of lines and conics, constructions of the first and second degrees, and so forth, are, after all, a very small part of mathematics, and they should be kept in their proper place.

I am afraid that I have wandered a considerable way from Prof. Hatton's book. It will be seen from what I have said that I am somewhat out of sympathy with its object, and that I regard a good deal of its undeniable ingenuity as misplaced. It is for this reason that I have not entered further into criticisms of detail, particularly with regard to Prof. Hatton's treatment of the trigonometry of the imaginary, about which a good deal might be said.

G. H. HARDY.

Th Elements of Analytical Conics. By C. DAVISON. Pp. 238. 10s. net. 1919. (Cambridge Univ. Press.)

Differential Calculus for Colleges and Secondary Schools. Pp. viii + 309. 6s. net. 1919. (Messrs. Bell & Sons.)

The "Elements" is a first course, omitting the general equation of the second degree and the higher parts of the subject. In addition to revision exercises and problem papers a very large number of questions are embodied in the text.

In what has been omitted from this book on the Calculus the author says that he has been guided by the needs of students in university colleges and of secondary school candidates for University scholarships. The examples are good: the revision exercises will be found useful: the essay papers will suggest to some enterprising teachers a means of making each pupil write his own text-book—or much of it: and there are a couple of dozen problem papers. The text is straightforwardly written, and is notable for the almost complete absence of any reference to the ideas upon which the calculus is based.

Decimal Tables. By Sir G. MOLESWORTH. Pp. 118. 2s. 6d. 1919. E. & F. N. Spon.)

This is an extremely handy little collection of tables, which will slip comfortably into the waistcoat pocket. The details under the heading "Length" are: Inch and foot scales, inches in five places converted into feet and yards to six places; inches to millimetres, advancing by 32nds; circumferences of circles advancing by 8ths and 10ths; circular arcs; f./s. to m./h. and conversely; decimals of feet in 16ths and 32nds; feet in links and conversely; nautical to statute miles and conversely. Similar tables follow for areas and for cubic measure, the latter containing earth-work tables. The "Weight" section contains wind-pressure, horse-power, atmospheric pressure, and heads of water. Tables of angular measurement, time, money, thermal units, short trigonometrical tables from 0° to 45°, logarithmic tables from 0 to 1000, squares, cubes, square and cube roots, of integers to 1000, and a table of reciprocals, are also given.

Solutions of the Examples in a Treatise on Differential Equations. By A. R. Forsyth. Pp. 249. 10s. net. 1918. (Macmillan.)

Professor Forsyth has conferred a boon on the teacher and on the private student by the publication of this volume. To the private student the advantage of a "key" to a book without "answers" is obvious enough, and its discreet use is to him an unmixed benefit. But the author of our classical treatises on the subject has had the teacher more than the needs of the private student in his mind in preparing this volume of solutions, as will be seen by the references supplied and by a glance at, for instance, pp. 68, 123. A few errors in the "Treatise" are corrected, and it is noted that with perhaps three exceptions all the equations set have proved soluble.

An Elementary Course of Infinitesimal Calculus. By H. Lamb. (Revised edition.) Pp. xiv+530. 20s. net. 1919. (Camb. Univ. Press.)

The new edition of Prof. Lamb's "Course" is welcome, in that it reflects the change in outlook of the experienced teacher during the last twenty years. Apart from alteration in detail, cases of omission and compression, there are two changes that stand out in relief, and these cannot be better described than in the author's own words:

"A special chapter is devoted to the exponential and allied functions, the exponential functions being now defined as the standard solution of the equation $\frac{dy}{dx}=y$. It is to this property, entirely, that the function owes its importance in mathematics, and it seems therefore most natural to take this as the starting point. No theory of the exponential series which has any pretensions to be rigorous can be said to be altogether elementary, but it is claimed that the method here followed is, from the standpoint of the Calculus, no more difficult than any other, whilst there can be no question as to its being the most appropriate. Another considerable change is in the treatment of infinite series, their differentiation and integration. In previous editions these questions were discussed in a general manner, by the light of the theory of uniform convergence. There was perhaps some justification for including this theory, at a time when it was hardly accessible in any English manual, but it was out of perspective with the rest of the book, and is now omitted. It is replaced by a discussion restricted to *power-series* only, which are the only type which the student is likely to be concerned with until he reaches a more advanced stage."

THE LIBRARY.

Change of Address.

The Library is now at 16 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

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Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them:

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
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Opere di Evangelista Torricelli. By G. Loria and G. Vassura. With Preface by G. Loria. Pp. xxxviii. 1919. (Montanari, Faenza.)

Les Spectres Numériques. By M. Petrovitch. Pp. viii + 110. 7 fr. 50 c. 1919. (Gauthier-Villars.)

Mathematical Papers for Admission into the Royal Military Academy, the Royal Military College, and Papers in Elementary Engineering for Naval Cadetships. March-July, 1919. Edited by R. M. Milne. Pp. 40. With Answers. 1s. 9d. net. 1919. (Macmillan.)

The Theory of the Imaginary in Geometry, together with the Trigonometry of the Imaginary. By J. L. S. Hatton. Pp. vi + 216. 18s. net. 1920. (Camb. Univ. Press.)

The Elements of Analytical Conics. By C. Davison. Pp. 238. 10s. net. 1919. (Camb. Univ. Press.)

Applied Aerodynamics. By L. Bairstow. Pp. xix + 565. 32s. net. 1920. (Longmans, Green & Co.)

The Theory of Determinants in the Historical Order of Development. By Sir Thomas Muir. Vol. III. *The period* 1861-1880. Pp. xxvi + 496. 35s. net. 1920. (Macmillan.)

Every-Day Mathematics. By F. Sandon. Pp. xii + 260. 4s. 6d. net. 1920. (Hodder & Stoughton.)

The Foundations of Einstein's Theory of Gravitation. By E. Freundlich. Translated by H. L. Brose, with Preface by H. H. Turner. Pp. xvi + 61. 5s. net. 1920. (Camb. Univ. Press.)

Tables Logarithmiques et Trigonométriques à quatre décimales . . . à l'usage des Physiciens et des Navigateurs. By H. R. Desvallées. Pp. xxiv + 72. 6 fr. 1920. (Gauthier-Villars.)

The Elementary Differential Geometry of Plane Curves. By R. H. Fowler. Pp. 105. 6s. net. (Camb. Univ. Press.)

Revista de Matematicas y fisicas elementales. Edited by Señor B. Ig. Baidaff. Vol. I. (Buenos Ayres. 1919.)

Leçons sur L'Approximation des Fonctions d'une Variable réelle. By C. de la Vallée Poussin. Pp. viii + 150. 8 fr. (temporarily raised to 12 fr.). 1919. (Gauthier-Villars.)

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A Linear Associative Algebra suitable for Electromagnetic Relations and the Theory of Relativity. Pp. 331-334. W. J. JOHNSTON. *On Generalized Relativity in connection with Mr. W. J. Johnston's Symbolic Calculus.* Pp. 334-363. SIR J. LARMOR.

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